

Calculus I

Chapter 3, Part 2

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Chapter 3. APPLICATIONS OF DIFFERENTIATION



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3.5 INDETERMINATE FORMS AND L'HÔPITAL'S RULE



3.5.1 INDETERMINATE QUOTIENTS

If we have a limit of the form

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$$

where both $f(x) \rightarrow 0$ and $g(x) \rightarrow 0$ as $x \rightarrow a$ then this limit **may or may not exist** and is called an **indeterminate form of type $\frac{0}{0}$** .

For instance, the following limits

$$\lim_{x \rightarrow 0} \frac{\sin x}{x^2}; \quad \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}; \quad \lim_{x \rightarrow 1} \frac{\ln(x)}{x^2 - 1}$$

are of the indeterminate form of type $\frac{0}{0}$.

3.5 INDETERMINATE FORMS AND L'HÔPITAL'S RULE



3.5.1 INDETERMINATE QUOTIENTS

If we have a limit of the form

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$$

where both $|f(x)| \rightarrow \infty$ and $|g(x)| \rightarrow \infty$, as $x \rightarrow a$, then the limit may or may not exist and is called an **indeterminate form of type $\frac{\infty}{\infty}$** .

3.5 INDETERMINATE FORMS AND L'HÔPITAL'S RULE

Or

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a). Suppose that

$$\lim_{x \rightarrow a} f(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = 0$$

or that
$$\lim_{x \rightarrow a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.) Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

3.5 INDETERMINATE FORMS AND L'HÔPITAL'S RULE

or

NOTES:

NOTE 1: L'Hospital's Rule says that the **limit of a quotient of functions is equal to** the **limit of the quotient of their derivatives**, provided that the given conditions are satisfied.

NOTE 2 • L'Hospital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity; that is, " $x \rightarrow a$ " can be replaced by any of the following symbols: $x \rightarrow a^+$, $x \rightarrow a^-$, $x \rightarrow \infty$, $x \rightarrow -\infty$.

3.5 INDETERMINATE FORMS AND L'HÔPITAL'S RULE



3.5.1 INDETERMINATE QUOTIENTS

CAUTION: It is especially important to verify the limit $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ is an indeterminate form of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ before using l'Hôpital's Rule.

3.5 INDETERMINATE FORMS AND L'HÔPITAL'S RULE

or

3.5.1 INDETERMINATE QUOTIENTS

CAUTION: It is especially important to verify the limit $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ is an indeterminate form of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ before using l'Hôpital's Rule.

EXAMPLE 1

Evaluate $\lim_{x \rightarrow 1} \frac{\ln x}{x^2 - 1}$.

Solution We have $\lim_{x \rightarrow 1} \frac{\ln x}{x^2 - 1} \quad \left[\frac{0}{0} \right]$

$$= \lim_{x \rightarrow 1} \frac{1/x}{2x} = \lim_{x \rightarrow 1} \frac{1}{2x^2} = \frac{1}{2}.$$

3.5 INDETERMINATE FORMS AND L'HÔPITAL'S RULE

Or

3.5.1 INDETERMINATE QUOTIENTS

EXAMPLE 2 Evaluate $\lim_{x \rightarrow 0} \frac{2 \sin x - \sin(2x)}{2e^x - 2 - 2x - x^2}$.

Solution We have (using l'Hôpital's Rule three times)

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{2 \sin x - \sin(2x)}{2e^x - 2 - 2x - x^2} & \quad \left[\frac{0}{0} \right] \\ &= \lim_{x \rightarrow 0} \frac{2 \cos x - 2 \cos(2x)}{2e^x - 2 - 2x} \quad \text{cancel the 2s} \\ &= \lim_{x \rightarrow 0} \frac{\cos x - \cos(2x)}{e^x - 1 - x} \quad \text{still} \quad \left[\frac{0}{0} \right] \\ &= \lim_{x \rightarrow 0} \frac{-\sin x + 2 \sin(2x)}{e^x - 1} \quad \text{still} \quad \left[\frac{0}{0} \right] \\ &= \lim_{x \rightarrow 0} \frac{-\cos x + 4 \cos(2x)}{e^x} = \frac{-1 + 4}{1} = 3. \end{aligned}$$

3.5 INDETERMINATE FORMS AND L'HÔPITAL'S RULE

or

3.5.1 INDETERMINATE QUOTIENTS

EXAMPLE 3

Evaluate (a) $\lim_{x \rightarrow (\pi/2)^-} \frac{2x - \pi}{\cos^2 x}$ and (b) $\lim_{x \rightarrow 1^+} \frac{x}{\ln x}$.

Solution

$$\begin{aligned} \text{(a)} \quad & \lim_{x \rightarrow (\pi/2)^-} \frac{2x - \pi}{\cos^2 x} \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &= \lim_{x \rightarrow (\pi/2)^-} \frac{2}{-2 \sin x \cos x} = -\infty \end{aligned}$$

3.5 INDETERMINATE FORMS AND L'HÔPITAL'S RULE

Cr

3.5.1 INDETERMINATE QUOTIENTS

- (b) l'Hôpital's Rule cannot be used to evaluate $\lim_{x \rightarrow 1+} x/(\ln x)$ because this is not an indeterminate form. The denominator approaches 0 as $x \rightarrow 1+$, but the numerator does not approach 0. Since $\ln x > 0$ for $x > 1$, we have, directly,

$$\lim_{x \rightarrow 1+} \frac{x}{\ln x} = \infty.$$

(Had we tried to apply l'Hôpital's Rule, we would have been led to the erroneous answer $\lim_{x \rightarrow 1+} (1/(1/x)) = 1$.)

BEWARE! Do not use l'Hopital's Rule to evaluate a limit that is not indeterminate.

3.5 INDETERMINATE FORMS AND L'HÔPITAL'S RULE



Example 5.1 Find the following limits

(a) $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x + x^2}.$

(b) $\lim_{x \rightarrow 0} \frac{x - \sin x}{x^3}.$

(c) $\lim_{x \rightarrow 0} \frac{\sin x}{x^2}.$

3.5 INDETERMINATE FORMS AND L'HÔPITAL'S RULE

3.5.2 INDETERMINATE PRODUCTS

The limit

$$\lim_{x \rightarrow a} f(x)g(x),$$

where $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} |g(x)| = \infty$, is called an **indeterminate form of type $0 \cdot \infty$** . By writing the product fg as a quotient

$$fg = \frac{f}{\frac{1}{g}} \quad \text{or} \quad fg = \frac{g}{\frac{1}{f}}$$

the given limit can be converted into an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$, so that we can use l'Hospital's Rule.

3.5 INDETERMINATE FORMS AND L'HÔPITAL'S RULE

or

3.5.2 INDETERMINATE PRODUCTS

Example:

$$(b) \lim_{x \rightarrow 0^+} x^a \ln x \quad (a > 0) \quad [0 \cdot (-\infty)]$$

$$= \lim_{x \rightarrow 0^+} \frac{\ln x}{x^{-a}} \quad \left[\frac{-\infty}{\infty} \right]$$

$$= \lim_{x \rightarrow 0^+} \frac{1/x}{-ax^{-a-1}} = \lim_{x \rightarrow 0^+} \frac{x^a}{-a} = 0.$$

3.5 INDETERMINATE FORMS AND L'HÔPITAL'S RULE

3.5.3 INDETERMINATE DIFFERENCES

If $\lim_{x \rightarrow a} f(x) = \infty$ and $\lim_{x \rightarrow a} g(x) = \infty$, then the limit

$$\lim_{x \rightarrow a} [f(x) - g(x)]$$

is called an **indeterminate form of type $\infty - \infty$** .

Example 5.3 Determine

$$\lim_{x \rightarrow 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right).$$

Example 5.4 Find

$$\lim_{x \rightarrow 0^+} \left(\frac{1}{x} - \frac{1}{\sin x} \right).$$

3.5 INDETERMINATE FORMS AND L'HÔPITAL'S RULE

or

EXAMPLE 4 Evaluate $\lim_{x \rightarrow 0+} \left(\frac{1}{x} - \frac{1}{\sin x} \right)$.

Solution The indeterminate form here is of type $[\infty - \infty]$ to which l'Hôpital's Rule cannot be applied. However, it becomes $[0/0]$ after we combine the fractions into one fraction.

$$\begin{aligned} & \lim_{x \rightarrow 0+} \left(\frac{1}{x} - \frac{1}{\sin x} \right) \quad [\infty - \infty] \\ &= \lim_{x \rightarrow 0+} \frac{\sin x - x}{x \sin x} \quad \left[\frac{0}{0} \right] \\ &= \lim_{x \rightarrow 0+} \frac{\cos x - 1}{\sin x + x \cos x} \quad \left[\frac{0}{0} \right] \\ &= \lim_{x \rightarrow 0+} \frac{-\sin x}{2 \cos x - x \sin x} = \frac{-0}{2} = 0. \end{aligned}$$

3.5 INDETERMINATE FORMS AND L'HÔPITAL'S RULE

or

3.5.4 INDETERMINATE POWERS



Indeterminate Powers

Several indeterminate forms arise from the limit

$$\lim_{x \rightarrow a} [f(x)]^{g(x)}$$

1. $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = 0$ type 0^0
2. $\lim_{x \rightarrow a} f(x) = \infty$ and $\lim_{x \rightarrow a} g(x) = 0$ type ∞^0
3. $\lim_{x \rightarrow a} f(x) = 1$ and $\lim_{x \rightarrow a} g(x) = \pm\infty$ type 1^∞

3.5 INDETERMINATE FORMS AND L'HÔPITAL'S RULE



3.5.4 INDETERMINATE POWERS

Limits of functions of the form

$$\lim_{x \rightarrow a} [f(x)]^{g(x)}$$

can lead to **indeterminate forms of the types 0^0 , ∞^0 or 1^∞** .

Each of these three cases **can be treated by taking the natural logarithm**:

$$\text{let } y = [f(x)]^{g(x)}, \quad \text{then } \ln y = g(x) \ln f(x)$$

In this method, we are led to the indeterminate product $g(x) \ln f(x)$, which is of type $0 \cdot \infty$.

3.5 INDETERMINATE FORMS AND L'HÔPITAL'S RULE



3.5.4 INDETERMINATE POWERS

Example 5.5 Evaluate

$$\lim_{x \rightarrow 0} (\cos x)^{\frac{1}{x^2}}.$$

Example 5.6 Find the limit

$$\lim_{x \rightarrow 0^+} (\cot x)^{\frac{1}{\ln x}}.$$

3.5 INDETERMINATE FORMS AND L'HÔPITAL'S RULE

3.5.5 L'HÔPITAL'S RULE FOR LIMITS AT INFINITY

Theorem 5.2 (L'Hôpital's Rule for Limits at Infinity)

Suppose that f and g are differentiable on an interval (b, ∞) and that $g'(x) \neq 0$ for $x > b$. If

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} g(x) = 0$$

or

$$\lim_{x \rightarrow \infty} |f(x)| = \lim_{x \rightarrow \infty} |g(x)| = \infty.$$

Then

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)}$$

provided that the limit on the right exists or is ∞ or $-\infty$. A similar result holds for limits as $x \rightarrow -\infty$.

3.5 INDETERMINATE FORMS AND L'HÔPITAL'S RULE



3.5.5 L'HÔPITAL'S RULE FOR LIMITS AT INFINITY

Example 5.7 Find the following limits

(a) $\lim_{x \rightarrow \infty} \frac{\ln x}{x^\alpha}, \quad \alpha > 0.$

(b) $\lim_{x \rightarrow \infty} \frac{x^n}{e^x}, \quad n \text{ is an integer.}$

Example 5.8 Evaluate $\lim_{x \rightarrow -\infty} \sqrt[3]{x^2} e^x$. Use the knowledge of this limit, together with information from derivatives, to sketch the curve $y = f(x) = \sqrt[3]{x^2} e^x$.

EXAMPLE 7 Evaluate $\lim_{x \rightarrow \infty} \left(1 + \sin \frac{3}{x}\right)^x$.

Solution This indeterminate form is of type 1^∞ . Let $y = \left(1 + \sin \frac{3}{x}\right)^x$. Then, taking $\ln$ of both sides,

$$\begin{aligned}\lim_{x \rightarrow \infty} \ln y &= \lim_{x \rightarrow \infty} x \ln \left(1 + \sin \frac{3}{x}\right) \quad [\infty \cdot 0] \\&= \lim_{x \rightarrow \infty} \frac{\ln \left(1 + \sin \frac{3}{x}\right)}{\frac{1}{x}} \quad \left[\frac{0}{0}\right] \\&= \lim_{x \rightarrow \infty} \frac{\frac{1}{1 + \sin \frac{3}{x}} \left(\cos \frac{3}{x}\right) \left(-\frac{3}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x \rightarrow \infty} \frac{3 \cos \frac{3}{x}}{1 + \sin \frac{3}{x}} = 3.\end{aligned}$$

Hence $\lim_{x \rightarrow \infty} \left(1 + \sin \frac{3}{x}\right)^x = e^3$.

3.5 INDETERMINATE FORMS AND L'HÔPITAL'S RULE

or

EXAMPLE 5 Evaluate (a) $\lim_{x \rightarrow \infty} \frac{x^2}{e^x}$ and (b) $\lim_{x \rightarrow 0+} x^a \ln x$, where $a > 0$.

$$\begin{aligned} \text{(a)} \quad & \lim_{x \rightarrow \infty} \frac{x^2}{e^x} \quad \left[\frac{\infty}{\infty} \right] \\ &= \lim_{x \rightarrow \infty} \frac{2x}{e^x} \quad \text{still } \left[\frac{\infty}{\infty} \right] \\ &= \lim_{x \rightarrow \infty} \frac{2}{e^x} = 0. \end{aligned}$$

Similarly, one can show that $\lim_{x \rightarrow \infty} x^n / e^x = 0$ for any positive integer n by repeated applications of l'Hôpital's Rule.

$$\begin{aligned} \text{(b)} \quad & \lim_{x \rightarrow 0+} x^a \ln x \quad (a > 0) \quad [0 \cdot (-\infty)] \\ &= \lim_{x \rightarrow 0+} \frac{\ln x}{x^{-a}} \quad \left[\frac{-\infty}{\infty} \right] \\ &= \lim_{x \rightarrow 0+} \frac{1/x}{-ax^{-a-1}} = \lim_{x \rightarrow 0+} \frac{x^a}{-a} = 0. \end{aligned}$$

3.6 OPTIMIZATION PROBLEMS



Example 6.1 An open rectangular box with square base is to be made from 48 ft^2 of material. What dimensions will result in a box with the largest possible volume ?

3.6 OPTIMIZATION PROBLEMS



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Solution: Let variable x be the length of one edge of the square base and variable y the height of the box.

The total surface area of the box is: (area of base) + 4 (area of one side)

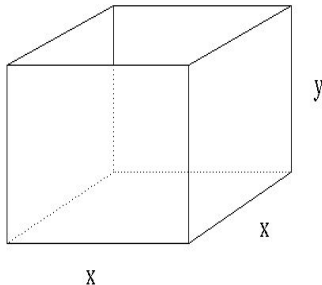
3.6 OPTIMIZATION PROBLEMS



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Solution: Let variable x be the length of one edge of the square base and variable y the height of the box.

The total surface area of the box is: (area of base) + 4 (area of one side) = $x^2 + 4(xy) = 48$.



The total surface area of the box is given to be

3.6 OPTIMIZATION PROBLEMS



It follows from $x^2 + 4(xy) = 48$ that $y = \frac{48-x^2}{4x}$.

We wish to MAXIMIZE the total VOLUME of the box

$$V = (\text{length}) (\text{width}) (\text{height}) = (x)(x)(y) = x^2y$$

3.6 OPTIMIZATION PROBLEMS



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This means that we want to find the maximum of the function:

$$V(x) = \frac{48x - x^3}{4}, \quad x > 0.$$

3.6 OPTIMIZATION PROBLEMS



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This means that we want to find the maximum of the function:

$$V(x) = \frac{48x - x^3}{4}, \quad x > 0.$$

We have

$$V'(x) = \frac{48 - 3x^2}{4} = 0 \quad (x > 0) \iff x = 4.$$

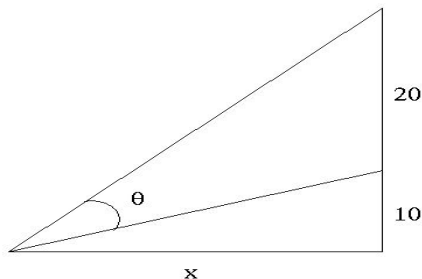
Moreover, $V'(x) > 0$ if $x < 4$ and $V'(x) < 0$ if $x > 4$. So $V(4) = 32$ is the maximum of V on $(0, \infty)$.

If $x=4$ ft. and $y=2$ ft. then $V = 32\text{ft}^3$ is the largest possible volume of the box.

3.6 OPTIMIZATION PROBLEMS

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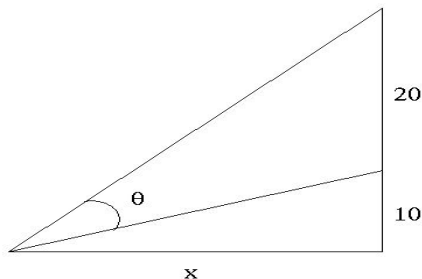
Example 6.2 A movie screen on a wall is 20 feet high and 10 feet above the floor. At what distance x from the front of the wall should you position yourself so that the viewing angle θ of the movie screen is as large as possible ? (See diagram).



3.6 OPTIMIZATION PROBLEMS



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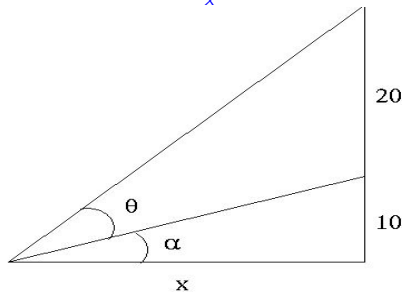
Solution: Let variable θ be the viewing angle and variable x the distance as denoted in the diagram.

3.6 OPTIMIZATION PROBLEMS



We seek to write angle θ as a function of distance x . Introduce angle α as in the diagram below.

We have $\tan \alpha = \frac{10}{x}$. Thus $\alpha = \arctan \frac{10}{x}$.

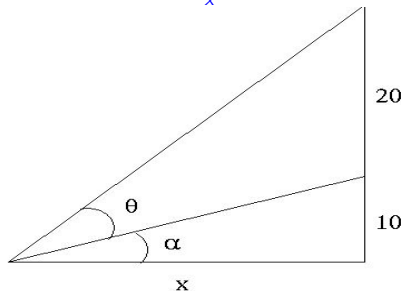


3.6 OPTIMIZATION PROBLEMS

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In a similar fashion:

$\tan(\theta + \alpha) = \frac{30}{x}$, so that $\theta + \alpha = \arctan \frac{30}{x}$. Therefore

$$\theta = \arctan \frac{30}{x} - \alpha = \arctan \frac{30}{x} - \arctan \frac{10}{x}, \quad x > 0.$$

3.6 OPTIMIZATION PROBLEMS



We want to find the maximum of the function

$$\theta(x) = \arctan \frac{30}{x} - \arctan \frac{10}{x}, \quad x > 0.$$

we have

$$\theta'(x) = \frac{-30}{x^2 + 30^2} + \frac{10}{x^2 + 10^2}.$$

Thus, $\theta'(x) = 0$ when $x = 10\sqrt{3}$. Moreover, $\theta'(x) > 0$ if $x < 10\sqrt{3}$ and $\theta'(x) < 0$ if $x > 10\sqrt{3}$. So

$$\theta(10\sqrt{3}) = \arctan \sqrt{3} - \arctan \frac{1}{\sqrt{3}} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}.$$

is the maximum value.

STEPS IN SOLVING OPTIMIZATION PROBLEMS

1. Read the problem very carefully until it is clearly understood what is given and what must be found.
2. Make a diagram if appropriate. In most problems it is useful to draw a diagram and identify the given and required quantities on the diagram.
3. Assign any symbols that are not already specified in the statement of the problem.
4. Express the quantity Q to be maximized or minimized as a function of one or more variables.
5. If Q is a function of more than one variable, use the information to find relationships (in the form of equations) among these variables to eliminate variables and hence express Q as a function of only one variable.
6. Find the required extremum value of the function Q .

3.7 NEWTON'S METHOD



Suppose that we need to find roots to the equation

$$f(x) = 0.$$

- There are only a few general classes of equations that we can solve exactly.
- For a quadratic equation $ax^2 + bx + c = 0$, there is a well-known formula for the roots.
- For third- and fourth-degree equations, there are also formulas for the roots. However, they are extremely complicated.
- For a polynomial of degree 5 or higher, there is no such formula.

3.7 NEWTON'S METHOD

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When exact formulae for solving an equation are not available, we can turn to numerical techniques to approximate the solutions we seek.



3.7 NEWTON'S METHOD

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When exact formulae for solving an equation are not available, we can turn to numerical techniques to approximate the solutions we seek.

♠ The goal of Newton's method for estimating a solution r of an equation $f(x) = 0$ is to produce a sequence of approximations that approach the solution.

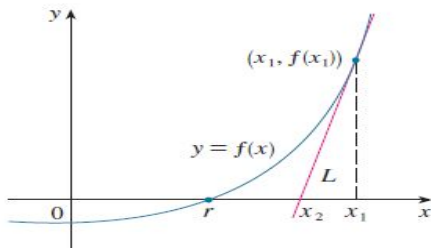
3.7 NEWTON'S METHOD

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Suppose that r is a solution of the equation $f(x) = 0$. However, we can not know the exact value of r . So we find an **approximate solution** to the equation $f(x) = 0$.

In fact, we find a sequence $x_1, x_2, \dots, x_n, \dots$ such that $\lim_{n \rightarrow \infty} x_n = r$.

We start with a first approximation x_1 , which is obtained by guessing, or from a rough sketch of the graph of f , or from a computer generated



graph of f . Then x_2 is the x-intercept of the tangent line to the curve $y = f(x)$ at $(x_1, f(x_1))$

3.7 NEWTON'S METHOD

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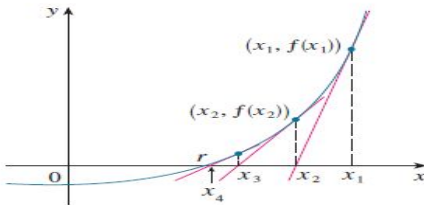
The equation of the tangent line at $(x_1, f(x_1))$ is

$y - f(x_1) = f'(x_1)(x - x_1)$. Thus,

$y(x_2) - f(x_1) = 0 - f(x_1) = f'(x_1)(x_2 - x_1)$. If $f'(x_1) \neq 0$, we can solve this equation for x_1 :

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

We use x_2 as a second approximation to r . Next we repeat this procedure



with x_1 replaced by x_2 .

x_3 in the x-intercept of the tangent line to the curve $y = f(x)$ at $(x_2, f(x_2))$.

Then

3.7 NEWTON'S METHOD

Cr

Then x_3 is given by

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}.$$

In general, if the n -th approximation is x_n and $f'(x_n) \neq 0$ then the next approximation is given by

2

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

If the numbers x_n become closer and closer to r as n becomes large, then we say that the sequence *converges* to r and we write

$$\lim_{n \rightarrow \infty} x_n = r$$

3.7 NEWTON'S METHOD

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The formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

is known as the **Newton's Method formula**.

- The procedure in going from n to $n + 1$ is the same for all values of n . It is called an **iterative process**.
- This method is known as **Newton's Method** or **The Newton-Raphson Method**.

3.7 NEWTON'S METHOD



NEWTON'S METHOD

To find a numerical approximation to a root of $f(x) = 0$:

1. Choose initial guess x_0 (close to the desired root if possible).
2. Use the first approximation x_0 to get a second, the second to get a third, and so on, using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad (f'(x_n) \neq 0)$$

where $f'(x_n)$ is the derivative of f at x_n .

3.7 NEWTON'S METHOD

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EXAMPLE 3

Use Newton's Method to find the only real root of the equation $x^3 - x - 1 = 0$ correct to 10 decimal places.

Solution We have $f(x) = x^3 - x - 1$ and $f'(x) = 3x^2 - 1$. Since f is continuous and since $f(1) = -1$ and $f(2) = 5$, the equation has a root in the interval $[1, 2]$. Figure 4.13 shows that the equation has only one root to the right of $x = 0$. Let us make the initial guess $x_0 = 1.5$. The Newton's Method formula here is

$$x_{n+1} = x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1} = \frac{2x_n^3 + 1}{3x_n^2 - 1},$$

so that, for example, the approximation x_1 is given by

$$x_1 = \frac{2(1.5)^3 + 1}{3(1.5)^2 - 1} \approx 1.347\,826\,\dots$$

3.7 NEWTON'S METHOD

Cr

Table 2.

n	x_n	$f(x_n)$
0	1.5	0.875 000 000 000 ...
1	1.347 826 086 96 ...	0.100 682 173 091 ...
2	1.325 200 398 95 ...	0.002 058 361 917 ...
3	1.324 718 174 00 ...	0.000 000 924 378 ...
4	1.324 717 957 24 ...	0.000 000 000 000 ...
5	1.324 717 957 24 ...	

The values in Table 2 were obtained with a scientific calculator. Evidently $r = 1.324\,717\,957\,2$ correctly rounded to 10 decimal places.

3.7 NEWTON'S METHOD

Cr

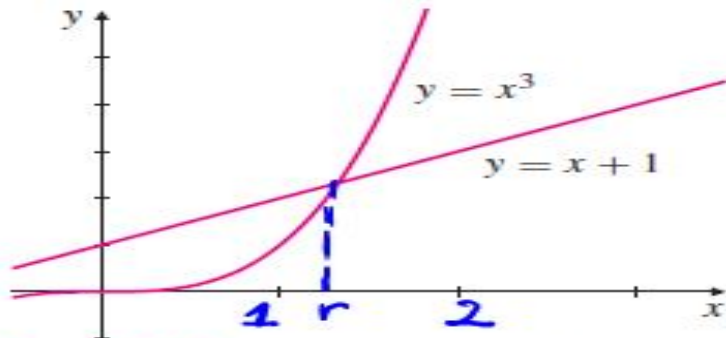


Figure 4.13 The graphs of x^3 and $x + 1$ meet only once to the right of $x = 0$, and that meeting is between 1 and 2

3.7 NEWTON'S METHOD

Cr

Example: Solve the equation $x^3 = \cos x$ to 11 decimal places.

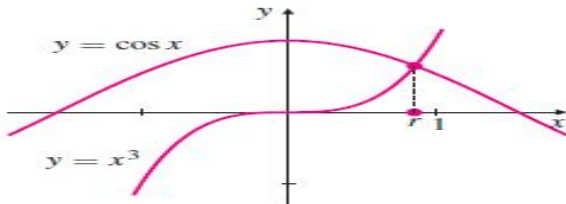


Figure 4.14 Solving $x^3 = \cos x$



3.7 NEWTON'S METHOD

Dr

Example: Solve the equation $x^3 = \cos x$ to 11 decimal places.

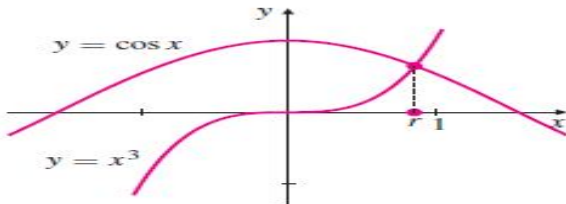


Figure 4.14 Solving $x^3 = \cos x$

♠ Solution It appears that the curves intersect slightly to the left of $x = 1$. So we choose $x_0 = 0.8$. If $f(x) = x^3 - \cos x$, then $f'(x) = 3x^2 + \sin x$. The Newton's Method formula for this function is

$$x_{n+1} = x_n - \frac{x_n^3 - \cos x_n}{3x_n^2 + \sin x_n} = \frac{2x_n^3 + x_n \sin x_n + \cos x_n}{3x_n^2 + \sin x_n}.$$

3.7 NEWTON'S METHOD



The approximations $x_1, x_2, \dots$, are given in the following table

n	x_n	$f(x_n)$
0	0.8	-0.184 706 709 347
1	0.870 034 801 135	0.013 782 078 762
2	0.865 494 102 425	0.000 006 038 051
3	0.865 474 033 493	0.000 000 001 176
4	0.865 474 033 102	0.000 000 000 000
5	0.865 474 033 102	0.000 000 000 000

Thus, the two curves intersect at $x = 0.865\ 474\ 033\ 102$, rounded to 11 decimal places.